On a Generalized Fan Inequality

Volker Mehrmann

Fakultät für Mathematik
Universität Bielefeld
Postfach 8640
4800 Bielefeld 1, Federal Republic of Germany

Submitted by Ludwig Elsner

ABSTRACT

It is shown that the ω- and τ-matrices, the weakly sign symmetric matrices, the R- and V-matrices, and the matrices c-equivalent to an M-matrix or to a real matrix with nonpositive off-diagonal elements, can all be characterized by the same determinantal inequality, which we call a generalized Fan inequality.

1. INTRODUCTION

It has been shown by Fan [6] that M-matrices satisfy several determinantal inequalities, which are generalized versions of the well-known Hadamard-Fischer inequality. We show that a further generalization of these inequalities allows a unified characterization of the classes $\omega_{\langle n\rangle}, \tau_{\langle n\rangle}$ (e.g. Engel and Schneider [5]), the weakly sign symmetric matrices (e.g. Kotelyanskiĭ [10]), the classes $R_{\langle n\rangle}, V_{\langle n\rangle}$ (e.g. Mehrmann [11]), and the M-matrices (or the real matrices) with nonpositive off-diagonal elements.

2. NOTATION AND DEFINITIONS

By $\mathbb{R}(\mathbb{C})$ we denote the real (complex) field; by $\mathbb{R}^{n,n}$ ($\mathbb{C}^{n,n}$) the real (complex) $n\times n$ matrices.

For a positive integer n, we set $\langle n\rangle := \{1,\dots,n\}$ and for $A\in\mathbb{C}^{n,n}$ and $\varnothing\subset\mu,\nu\subseteq\langle n\rangle$, we denote by $A[\mu]$ the matrix $[a_{ij}]$, with $i,j\in\mu$ ($A[\mu]\in$

52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

$\mathbb{C}^{|\mu|,|\mu|}$, where $|\mu|$ denotes the cardinality of μ), and by $A[\mu|\nu]$ the matrix $[a_{ij}]$, $i \in \mu$, $j \in \nu$. For $A \in \mathbb{C}^{n,n}$, $\varnothing \subset \mu \subseteq \langle n \rangle$, and $t \in \mathbb{R}$, we set $A_t[\mu] := (A - tI)[\mu]$, where I denotes the identity matrix. We denote by $\sigma(A)$ the spectrum of A, and by $\rho(A) := \max\{|\lambda| \,|\, \lambda \in \sigma(A)\}$ the spectral radius of A, and furthermore we set

$$l(A) := \begin{cases} \min(\sigma(A) \cap \mathbb{R}) & \text{if } \sigma(A) \cap \mathbb{R} \neq \varnothing, \\ \infty & \text{otherwise.} \end{cases}$$

DEFINITION 1. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called

(1) a *Z-matrix* ($A \in Z_{\langle n \rangle}$) if $A = \alpha I - B$, with $\alpha \in \mathbb{R}$ and B a nonnegative matrix ($B \geqslant 0$),

(2) an *M-matrix* ($A \in M_{\langle n \rangle}$) if $A = \alpha I - B$ with $B \geqslant 0$ and $\alpha \geqslant \rho(B)$,

(3) *weakly sign symmetric* ($A \in K_{\langle n \rangle}$) if all principal minors of A are real and

$$\det A[\mu|\nu] \det A[\nu|\mu] \geqslant 0$$

$$\forall \mu, \nu \subseteq \langle n \rangle, \quad \text{with } 0 \neq |\mu| = |\nu| = |\mu \cup \nu| - 1, \tag{2.1}$$

(4) an *ω-matrix* ($A \in \omega_{\langle n \rangle}$) if $l(A[\mu]) < \infty$ $\forall \mu \subseteq \langle n \rangle$, $\mu \neq \varnothing$, and $l(A[\mu]) \leqslant l(A[\nu])$ $\forall \nu \subseteq \mu$, $\mu, \nu \neq \varnothing$,

(5) a *τ-matrix* ($A \in \tau_{\langle n \rangle}$) if $A \in \omega_{\langle n \rangle}$ and $l(A[\langle n \rangle]) \geqslant 0$,

(6) an *R-matrix* ($A \in R_{\langle n \rangle}$) if for all $\mu, \nu \subseteq \langle n \rangle$, $\mu, \nu \neq \varnothing$, and all $t \in \mathbb{R}$ such that all principal minors of $A_t[\mu \cup \nu]$ are nonnegative,

$$\det A_t[\mu] \det A_t[\nu] \geqslant \det A_t[\mu \cup \nu] \det A_t[\mu \cap \nu] \tag{2.2}$$

(we always use the convention $\det A[\varnothing] := 1$),

(7) a *V-matrix* ($A \in V_{\langle n \rangle}$) if $A \in R_{\langle n \rangle}$ and

$$\det A[\mu] \geqslant 0 \qquad \forall \mu \subseteq \langle n \rangle.$$

Following Engel and Schneider [4], we give the following definition:

DEFINITION 2.

(a) Let n be an integer, $n \geqslant 2$. A sequence of distinct integers $(i_1, \ldots, i_k)$ with $i_l \in \langle n \rangle$ $\forall l = 1, \ldots, k$, $k \geqslant 2$, is called a *cycle*. If $k = n$, then the cycle is called *full*. The set of all full cycles in a set of integers μ is denoted by Γ_μ. For $A=[a_{ij}]\in\mathbb{C}^{n,n}$ and a cycle $\gamma=(i_1,\dots,i_k)$, we call $\Pi_\gamma(A)=a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$ a *cyclic product*.

(b) For $A, B\in\mathbb{C}^{n,n}$, $A=[a_{ij}]$ and $B=[b_{ij}]$, we say $A \underset{c}{\sim} B$ (A *c-equivalent to* B) if

$$\Pi_\gamma(A)=\Pi_\gamma(B) \qquad \text{for all cycles } \gamma \text{ in } \langle n\rangle, \tag{i}$$

$$a_{ii}=b_{ii} \qquad \forall i\in\langle n\rangle. \tag{ii}$$

(c) For a class of matrices $A_{\langle n\rangle}\subseteq\mathbb{C}^{n,n}$ we set

$$A^c_{\langle n\rangle}:=\left\{A\in\mathbb{C}^{n,n}\,|\,\exists B\in A_{\langle n\rangle} \text{ with } B\underset{c}{\sim}A\right\}.$$

3. CHARACTERIZATION OF CLASSES OF MATRICES BY THE GENERALIZED FAN INEQUALITY

Although at first glance the classes of matrices defined in Section 2 look quite different and the relationship is not clear offhand, there exist several results, by different authors, indicating the relationship (e.g. Engel and Schneider [5], Carlson [3], Gantmacher and Krein [7]). A survey of results on the above classes, in particular the classes $R_{\langle n\rangle}, V_{\langle n\rangle}$ is given in Mehrmann [11]. The main purpose of this paper is to characterize (and classify) all the above classes by the same determinantal inequality, which we give as follows: Let $A\in\mathbb{C}^{n,n}$ have all principal minors real. Let $\mu_1,\dots,\mu_k$, $k\geqslant 2$, be a collection of k subsets of $\langle n\rangle$, and let $U\subseteq\mathbb{R}$. Consider the following *generalized Fan inequality* for all $t\in U$:

$$\left(\prod_{\substack{\alpha\subseteq\langle k\rangle\\ |\alpha|\text{ even}}}\det A_t\left[\bigcap_{i\in\alpha}\mu_i\right]\right)\cdot\det A_t\left[\bigcup_{i=1}^{k}\mu_i\right]\leqslant\prod_{\substack{\alpha\subseteq\langle k\rangle\\ |\alpha|\text{ odd}}}\det A_t\left[\bigcap_{i\in\alpha}\mu_i\right]. \tag{3.1}$$

By specifying k, $\mu_1,\dots,\mu_k$, and U in (3.1), we can now characterize all classes (1)–(7) defined in Section 2. The characterization of $\omega_{\langle n\rangle}$ is due to Engel and Schneider [5, p. 163].

THEOREM 1. *Let $A \in \mathbb{C}^{n,n}$. Then $A \in \omega_{\langle n \rangle}$ iff A has all principal minors real, and the generalized Fan inequality* (3.1) *holds for any* two disjoint *subsets $\mu_1, \mu_2 \subseteq \langle n \rangle$ and all $t \in U = \{t \in \mathbb{R} \mid$ all principal minors of $A_t[\mu_1 \cup \mu_2]$ are nonnegative$\}$.*

For $K_{\langle n \rangle}$ the characterization is due to Carlson [3, p. 125], and in slightly stronger forms to Hershkowitz and Berman [9] and Green [8, p. 356].

THEOREM 2. *Let $A \in \mathbb{C}^{n,n}$ have all principal minors real and positive. Then $A \in K_{\langle n \rangle}$ iff the generalized Fan inequality* (3.1) *holds for any* two subsets *of $\langle n \rangle$ and all $t \in U = \{0\}$.*

By definition, $A \in R_{\langle n \rangle}$ iff (3.1) holds for any *two* subsets $\mu_1, \mu_2 \subseteq \langle n \rangle$ and all $t \in U = \{t \in \mathbb{R} \mid$ all principal minors of $A_t[\mu_1 \cup \mu_2]$ are nonnegative$\}$.

The classes $\omega_{\langle n \rangle}, R_{\langle n \rangle}, K^+_{\langle n \rangle} = \{A \in K_{\langle n \rangle} \mid$all principal minors of A are positive$\}$ are invariant under the relation $\underset{c}{\sim}$, but $Z_{\langle n \rangle}$ is not. Thus, since we only consider determinants here, we characterize $Z^c_{\langle n \rangle}$ instead of $Z_{\langle n \rangle}$.

THEOREM 3. *Let $A \in \mathbb{C}^{n,n}$. Then the following are equivalent:*

(*i*) *All principal minors of A are real and*

$$(-1)^{|\mu|+1} \sum_{\gamma \in \Gamma_\mu} \Pi_\gamma(A) \leqslant 0 \qquad \forall \mu \subseteq \langle n \rangle, \quad |\mu| > 1. \tag{3.2}$$

(*ii*) $A \in Z^c_{\langle n \rangle}$.

(*iii*) *All principal minors of A are real, and for any finite collection of subsets of $\langle n \rangle$, $\mu_1, \dots, \mu_k$, $k \geqslant 2$, and all $t \in U = \{t \in \mathbb{R} \mid$ all principal minors of $A_t[\bigcup_{i=1}^k \mu_i]$ are nonnegative$\}$, the generalized Fan inequality* (3.1) *holds.*

Proof. (i)$\Leftrightarrow$(ii): This part of the proof is given by Theorem 6.6 in Engel and Schneider [5, p. 173].

(ii)$\Rightarrow$(iii): By a result of Fan [6, p. 47], we have that for any $A \in M_{\langle n \rangle}$ and for any finite collection of subsets of $\langle n \rangle$ and for $t = 0$, (3.1) holds. Let $A \in Z^c_{\langle n \rangle}$, $\mu_1, \dots, \mu_k \subseteq \langle n \rangle$, where $k \geqslant 2$, and let $\delta := \bigcup_{i=1}^k \mu_i$. If $t \in \mathbb{R}$ is such that all principal minors of $A_t[\delta]$ are nonnegative, then $A_t[\delta] \in M^c_{\langle |\delta| \rangle}$. Hence, there exists $B \in M_{\langle |\delta| \rangle}$ with $B \underset{c}{\sim} A_t[\delta]$. Thus (3.1) holds for B, and we have (iii).

(iii)$\Rightarrow$(ii): Assume (iii) holds, and define the following two functions: For $\mu \subseteq \langle n \rangle$ and $\nu \subseteq \mu$ let

$$\phi^N(A_t, \mu, \nu) = \prod_{\substack{\alpha \subseteq \nu \\ |\alpha| \text{ even}}} \det A_t[\mu \setminus \alpha]$$

and

$$\phi^D(A_t,\mu,\nu)=\prod_{\substack{\alpha\subseteq\nu\\|\alpha|\ \mathrm{odd}}}\det A_t[\mu\setminus\alpha].$$

Consider the following collection of subsets of $\langle n\rangle$: $\mu_j=\mu\setminus\{i_j\}$ for $j=1,\dots,k$, where $\nu=\{i_1,\dots,i_k\}$. Then (iii) implies that for all $t\in\mathbb{R}$ such that all principal minors of $A_t[\mu]$ are nonnegative,

$$\phi^N(A_t,\mu,\nu)\leqslant\phi^D(A_t,\mu,\nu). \tag{3.3}$$

Define the polynomial

$$P_{\mu,\nu}(t):=\phi^D(A_t,\mu,\nu)-\phi^N(A_t,\mu,\nu)=:\sum_{i=0}^{q}\alpha_i(-t)^{q-i}.$$

We will show that for all $\mu,\nu\subseteq\langle n\rangle$ with $\nu\subseteq\mu$, $P_{\mu,\nu}(t)$ is a polynomial of degree at most

$$q:=\left(\sum_{\substack{s=0\\ s\ \mathrm{even}}}^{|\nu|}\left[\binom{|\nu|}{s}(|\mu|-s)\right]\right)-|\nu|$$

and the coefficient of $(-t)^q$ is $(-1)^{|\nu|+2}\sum_{\gamma\in\Gamma_\nu}\Pi_\gamma(A)$. Then (iii) implies that for $t\to-\infty$, $P_{\mu,\nu}(t)\to\infty$. Hence, for $\mu=\nu$ we get $(-1)^{|\mu|+2}\sum_{\gamma\in\Gamma_\mu}\Pi_\gamma(A)\geqslant0$ and therefore we have (ii).

We show by induction that $P_{\mu,\nu}$ has the stated form. First, we show that for $n\geqslant2$ and all $\nu\subseteq\mu=\langle n\rangle$ with $|\nu|=2$, $P_{\mu,\nu}$ is of degree at most $2n-4$ and the coefficient of $(-t)^{2n-4}$ is $(-1)^4\sum_{\gamma\in\Gamma_\nu}\Pi_\gamma(A)$.

Let $\nu=\{i,j\}$, $i\neq j$. Then,

$$\begin{aligned}P_{\mu,\nu}(t)&=\phi^D(A_t,\mu,\nu)-\phi^N(A_t,\mu,\nu)\\&=\det A_t[\mu\setminus\{i\}]\det A_t[\mu\setminus\{j\}]-\det A_t[\mu]\det A_t[\mu\setminus\{i,j\}]\\&=(-t)^{2n-4}\big(a_{ii}a_{jj}-\det A[\{i,j\}]\big)+O(t^{2n-5})\qquad(\text{as}\quad t\to\infty)\\&=(-t)^{2n-4}(-1)^{2n-4}a_{ij}a_{ji}+O(t^{2n-5})\\&=(-t)^{2n-4}(-1)^{|\nu|+2}\sum_{\gamma\in\Gamma_\nu}\Pi_\gamma(A),\end{aligned}\tag{3.4}$$

since

$$\left[\sum_{\substack{s=0\\ s \text{ even}}}^{2} \binom{2}{s}(n-s)\right]-2=2n-4.$$

Now assume that $\forall \mu \subseteq \langle n \rangle$, $|\mu| = k > 2$, and $\forall \nu \subseteq \mu$, $|\nu| = j \leqslant k-1$, $P_{\mu,\nu}(t)$ is a polynomial of degree at most

$$q:=\left[\sum_{\substack{s=0\\ s \text{ even}}}^{|\nu|} \binom{|\nu|}{s}(|\mu|-s)\right]-|\nu|,$$

and the coefficient of $(-t)^q$ is $(-1)^{|\nu|+2}\sum_{\gamma \in \Gamma_\nu} \Pi_\gamma(A)$. Then consider $\mu \subseteq \langle n \rangle$, $|\mu| = k$, $\tilde{\nu} \subseteq \mu$, $|\tilde{\nu}| = j+1$, $\tilde{\nu} \setminus \nu = \{p\}$. We have

$$\begin{aligned}
P_{\mu,\tilde{\nu}}(t) &= \phi^D(A_t,\mu,\tilde{\nu}) - \phi^N(A_t,\mu,\tilde{\nu}) \\
&= \prod_{\substack{\alpha \subseteq \tilde{\nu}\\ |\alpha| \text{ odd}}} \det A_t[\mu \setminus \alpha] - \prod_{\substack{\alpha \subseteq \tilde{\nu}\\ |\alpha| \text{ even}}} \det A_t[\mu \setminus \alpha] \\
&= \prod_{\substack{\alpha \subseteq \nu\\ |\alpha| \text{ odd}}} \det A_t[\mu \setminus \alpha] \cdot \prod_{\substack{\alpha \subseteq \tilde{\nu}\\ |\alpha| \text{ odd}\\ p \in \alpha}} \det A_t[\mu \setminus \alpha] \\
&\quad - \prod_{\substack{\alpha \subseteq \nu\\ |\alpha| \text{ even}}} \det A_t[\mu \setminus \alpha] \cdot \prod_{\substack{\alpha \subseteq \tilde{\nu}\\ |\alpha| \text{ even}\\ p \in \alpha}} \det A_t[\mu \setminus \alpha] \\
&= \phi^D(A_t,\mu,\nu)\phi^N(A_t,\mu \setminus \{p\},\nu) - \phi^D(A_t,\mu \setminus \{p\},\nu)\phi^N(A_t,\mu,\nu) \\
&= \phi^N(A_t,\mu \setminus \{p\},\nu)P_{\mu,\nu}(t) - \phi^N(A_t,\mu,\nu)P_{\mu \setminus \{p\},\nu}(t).
\end{aligned} \tag{3.5}$$

By our inductive assumption, we have that degree $P_{\mu,\nu}(t)$ is at most

$$\left[\sum_{\substack{s=0\\ s \text{ even}}}^{|\nu|} \binom{|\nu|}{s}(|\mu|-s)\right]-|\nu| = \left[\sum_{\substack{s=0\\ s \text{ even}}}^{j} \binom{j}{s}(k-s)\right]-j,$$

and degree $P_{\mu\setminus\{p\},\nu}(t)$ is at most

$$\left[\sum_{\substack{s=0\\ s\text{ even}}}^{j}\binom{j}{s}(k-s-1)\right]-j.$$

Similarly, degree $\phi^N(A_t,\mu,\nu)$ is at most

$$\sum_{\substack{s=0\\ s\text{ even}}}^{j}\binom{j}{s}(k-s),$$

and degree $\phi^N(A_t,\mu\setminus\{p\},\nu)$ is at most

$$\sum_{\substack{s=0\\ s\text{ even}}}^{j}\binom{j}{s}(k-s-1).$$

Therefore, degree $P_{\mu,\tilde{\nu}}(t)$ is at most

$$\left[\sum_{s=0}^{j}\binom{j}{s}(k-s)\right]+\left[\sum_{\substack{s=0\\ s\text{ even}}}^{j}\binom{j}{s}(k-s-1)\right]-j$$

$$=\left[\sum_{\substack{s=0\\ s\text{ even}}}^{j+1}\binom{j+1}{s}(k-s)\right]-j:=q_1.$$

By our inductive assumption we have furthermore that the highest coefficients of $P_{\mu,\nu}(t)$ and $P_{\mu\setminus\{p\},\nu}(t)$ are equal. The highest coefficients of $\phi^N(A_t,\mu,\nu)$ and $\phi^D(A_t,\mu,\nu)$ are 1. Hence, $P_{\mu,\tilde{\nu}}(t)$ is a polynomial of degree at most

$$q_1-1=\left[\sum_{\substack{s=0\\ s\text{ even}}}^{j+1}\binom{j+1}{s}(k-s)\right]-j-1.$$

It remains to show that the coefficient of $(-t)^{q_1-1}$ is $(-1)^{|\tilde{\nu}|+2}\sum_{\gamma\in\Gamma_{\tilde{\nu}}}\Pi_\gamma(A)$.

Observe that the coefficient of $(-t)^{q_1-1}$ remains the same if we set all diagonal elements of A equal to 0. To see this, consider $\tilde{A}=A-\operatorname{diag}(a_{ii})$.

Then,

$$\begin{aligned}
&\phi^D(\tilde{A}_t,\mu,\tilde{\nu})-\phi^N(\tilde{A}_t,\mu,\tilde{\nu})-\phi^D(A_t,\mu,\nu)+\phi^N(A_t,\mu,\tilde{\nu})\\
&\quad=\phi^N(A_t,\mu,\nu)\big[\phi^D(A_t,\mu\setminus\{p\},\nu)-\phi^N(A_t,\mu\setminus\{p\},\nu)\\
&\qquad-\phi^D(\tilde{A}_t,\mu\setminus\{p\},\nu)+\phi^N(\tilde{A}_t,\mu\setminus\{p\},\nu)\big]\\
&\qquad+\big[\phi^N(A_t,\mu,\nu)-\phi^N(\tilde{A}_t,\mu,\nu)\big]\\
&\qquad\times\big[\phi^D(\tilde{A}_t,\mu\setminus\{p\},\nu)-\phi^N(\tilde{A}_t,\mu\setminus\{p\},\nu)\big]\\
&\qquad-\phi^N(A_t,\mu\setminus\{p\},\nu)\\
&\qquad\times\big[\phi^D(A_t,\mu,\nu)-\phi^N(A_t,\mu,\nu)-\phi^D(\tilde{A}_t,\mu,\nu)+\phi^N(\tilde{A}_t,\mu,\nu)\big]\\
&\qquad-\big[\phi^N(A_t,\mu\setminus\{p\},\nu\big]\\
&\qquad-\phi^N\big[\tilde{A}_t,\mu\setminus\{p\},\nu)\big]\big[\phi^D(\tilde{A}_t,\mu,\nu)-\phi^N(\tilde{A}_t,\mu,\nu)\big].
\end{aligned}$$

This follows by using (3.5). By our inductive assumption, each of these four terms has a degree at most

$$\left[\sum_{\substack{s=0\\ s\text{ even}}}^{|\nu|+1}\binom{|\nu|+1}{s}(|\mu|-s)\right]-(|\nu|+1)-1=q_1-2.$$

Hence, the sum of the four products has the same degree, too. Therefore, the coefficient of $(-t)^{q_1-1}$ contains only products of off-diagonal elements. Still, there might be products of cyclic products of order less than $|\nu|+1$, such that the order of the product is $|\nu|+1$. However, this cannot happen, as we show by a combinatorial argument.

Let $(\alpha_1,\ldots,\alpha_l)$ be an l-tuple of subsets of μ, with $\alpha_i\subset\mu$ and $|\alpha_i|=k_i$ $\forall i=1,\ldots,l$, and let $\sum_{i=1}^{l}k_i=|\tilde{\nu}|=j+1$. Only products of the form $\Pi_{\gamma_1}(A)\cdots\Pi_{\gamma_l}(A)$ with $\gamma_i\in\Gamma_{\alpha_i}$ $\forall i=1,\ldots,l$ can occur in the coefficient of $(-t)^{q_1-1}$ in $P_{\mu,\tilde{\nu}}(t)$. We show that, except for the case $l=1$ and $\alpha_1=\tilde{\nu}$, all these possible products occur equally often in $\phi^D(A_t,\mu,\tilde{\nu})$ and $\phi^N(A_t,\mu,\tilde{\nu})$, with the same sign. Therefore these products vanish when we take $\phi^D-\phi^N$. This implies that the coefficient of $(-t)^{q_1-1}$ is $-(-1)^{|\tilde{\nu}|+1}\sum_{\gamma\in\Gamma_{\tilde{\nu}}}\Pi_\gamma(A)$, since these products occur only in ϕ^N.

We have to count now how often a tuple $(\alpha_1,\ldots,\alpha_l)$ occurs, i.e., how often there are sets $\mu \setminus \beta_i$ in ϕ^N and ϕ^D such that $\beta_i \neq \beta_j$ $\forall i \neq j$, $i, j \in \{1,\ldots,l\}$, and $(\mu \setminus \beta_i) \supset \alpha_i$ $\forall i = 1,\ldots,l$. For any $\alpha_i \subseteq \mu$ we have

$$\phi^N(A_t,\mu,\nu) = \prod_{\substack{\alpha \subseteq \tilde{\nu} \\ |\alpha| \text{ even} \\ \alpha_i \subseteq \alpha}} \det A_t[\mu \setminus \alpha] \cdot \prod_{\substack{\alpha \subseteq \tilde{\nu} \\ |\alpha| \text{ even} \\ \varnothing \neq |\alpha_i \cap \alpha| \neq k_i}} \det A_t[\mu \setminus \alpha]$$

$$\cdot \prod_{\substack{\alpha \subseteq \tilde{\nu} \\ |\alpha| \text{ even} \\ \alpha_i \cap \alpha = \varnothing}} \det A_t[\mu \setminus \alpha].$$

The cyclic products of order $|\alpha_i|$ in α_i can only occur in those determinants $\det A_t[\mu \setminus \alpha]$ with $\alpha_i \cap \alpha = \varnothing$. Now let $q_i := |\alpha_i \cap \tilde{\nu}|$. Then there are

$$\sum_{\substack{s=0 \\ s \text{ even}}}^{j+1-q_i} \binom{j+1-q_i}{j+1-q_i-s} = \sum_{\substack{s=0 \\ s \text{ even}}}^{j+1} \binom{j+1-q_i}{s}$$

subsets $\mu \setminus \alpha$ of μ with $\alpha \subseteq \tilde{\nu}$, $|\alpha|$ even, $\alpha \supseteq \alpha_i$. Similarly for ϕ^D, we have that there are

$$\sum_{\substack{s=1 \\ s \text{ odd}}}^{j+1} \binom{j+1-q_i}{s}$$

subsets $\mu \setminus \alpha$ of μ with $\alpha \subseteq \tilde{\nu}$, $|\alpha|$ odd, $\alpha_i \subseteq \alpha$. But

$$\sum_{\substack{s=1 \\ s \text{ odd}}}^{j+1} \binom{j+1-q_i}{s} = \sum_{\substack{s=0 \\ s \text{ even}}}^{j+1} \binom{j+1-q_i}{s}.$$

Let now

$$n_\beta^e = \#\text{ subsets } \mu \setminus \alpha \text{ of } \mu,\ \alpha \subseteq \tilde{\nu},\ |\alpha| \text{ even},\ \beta \subseteq \mu \setminus \alpha,$$

$$n_\beta^o = \#\text{ subsets } \mu \setminus \alpha \text{ of } \mu,\ \alpha \subseteq \tilde{\nu},\ |\alpha| \text{ odd},\ \beta \subseteq \mu \setminus \alpha,$$

and for $\eta = \{j_1,\ldots,j_r\} \subseteq \{1,\ldots,l\}$ let $V_\eta := \bigcup_{i=1}^r \alpha_{j_i}$. Then, the number of times a product $\Pi_{\gamma_1}(A),\ldots,\Pi_{\gamma_l}(A)$ occurs in ϕ^N is $\sum_{\eta \subseteq \{1,\ldots,l\}} (-1)^{|\eta|+1} n_{V_\eta}^e$, and similarly for ϕ^D the number is $\sum_{\eta \subseteq \{1,\dots,l\}}(-1)^{|\eta|+1} n^o_{V_\eta}$, by a usual inclusion-exclusion argument. We have seen above that $n^e_{V_\eta} = n^o_{V_\eta}$. Thus, we have that the highest coefficient of $P_{\mu,\bar{\nu}}$ is of the form wanted. By induction, this result then holds for all $\mu \subseteq \langle n \rangle$ and all $\nu \subseteq \mu$. ■

We have not characterized $\tau_{\langle n \rangle}, \nu_{\langle n \rangle}, M^c_{\langle n \rangle}$ by (3.1) yet, but this is always a simple corollary of the corresponding results on $\omega_{\langle n \rangle}, R_{\langle n \rangle}, Z^c_{\langle n \rangle}$.

COROLLARY 1. *Let $A \in \mathbf{C}^{n,n}$. Then $A \in \tau_{\langle n \rangle}$ iff all principal minors of A are real and the generalized Fan inequality* (3.1) *holds for any* two disjoint *$\mu_1, \mu_2 \subseteq \langle n \rangle$ and all $t \in U = \{t \in \mathbb{R} \mid$ all principal minors of $A_t[\mu_1 \cup \mu_2]$ are nonnegative$\}$ and $0 \in U$.*

Proof. The proof follows directly by Theorem 1 above and Theorem 3.6 of Engel and Schneider [5, p. 161], where it is shown that for $A \in \omega_{\langle n \rangle}$ the following are equivalent:

(i) $A \in \tau_{\langle n \rangle}$,
(ii) $\det A[\mu] \geqslant 0$ for all $\mu \subseteq \langle n \rangle$, $\mu \neq \varnothing$. ■

COROLLARY 2. *Let $A \in \mathbf{C}^{n,n}$. Then $A \in M^c_{\langle n \rangle}$ iff all principal minors of A are real and the generalized Fan inequality* (3.1) *holds for any finite collection of subsets of $\langle n \rangle, \mu_1, \dots, \mu_k$ $k \geqslant 2$, and all $t \in U = \{t \in \mathbb{R} \mid$ all principal minors of $A_t[\bigcup_{i=1}^k \mu_i]$ are nonnegative$\}$ and $0 \in U$.*

Proof. The proof is clear by Theorem 3 and the fact that $A \in M_{\langle n \rangle}$ iff $A \in Z_{\langle n \rangle}$ and all principal minors of A are nonnegative (e.g., Berman and Plemmons [2, p. 149]). ■

The characterization of $V_{\langle n \rangle}$ is given by definition.
Another immediate consequence of the above results is:

COROLLARY 3.

(i) $M_{\langle n \rangle} \subset M^c_{\langle n \rangle} \subset V_{\langle n \rangle} = V^c_{\langle n \rangle} \subset \tau_{\langle n \rangle} \cap K_{\langle n \rangle} = (\tau_{\langle n \rangle} \cap K_{\langle n \rangle})^c$
(ii) $Z_{\langle n \rangle} \subset Z^c_{\langle n \rangle} \subset R_{\langle n \rangle} = R^c_{\langle n \rangle} \subset \omega_{\langle n \rangle} = \omega^c_{\langle n \rangle}$.

Proof. All inclusions follow directly by the above result except for the case $V_{\langle n \rangle} \subset \tau_{\langle n \rangle} \cap K_{\langle n \rangle}$. This follows by Theorem 2 in Hershkowitz and Berman [9]. ■

4. CONCLUDING REMARKS

As we have shown in Section 3, there is a unifying characterization of the classes defined in Section 2. Unfortunately the condition (3.1) is very hard to check for an arbitrary matrix $A \in \mathbb{C}^{n,n}$. It would be very helpful to have a correspondence between (3.1) and eigenvalue properties like those defining $\tau_{\langle n \rangle}$. A partial result in this direction, a necessary condition on the smallest real eigenvalues of the principal submatrices of $A \in V_{\langle n \rangle}$, will be given in a subsequent paper. (See also [11].)

REFERENCES

1 E. F. Beckenbach and R. Belman, *Inequalities*, Ergeb. Math. Grenzgeb., Springer, Berlin, 1961.
2 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.
3 D. Carlson, Weakly sign symmetric matrices and some determinantal inequalities, *Colloq. Math.* 17:123–129 (1967).
4 G. M. Engel and H. Schneider, Cyclic and diagonal products on a matrix, *Linear Algebra Appl.* 7:301–335 (1973).
5 G. M. Engel and H. Schneider, The Hadamard/Fischer inequality for a class of matrices defined by eigenvalue monotonicity, *Linear and Multilinear Algebra* 4:155–176 (1976).
6 K. Fan, Note on *M*-matrices, *Quart. J. Math. Oxford Ser.* (2) 11:43–49 (1960).
7 F. R. Gantmacher and M. G. Krein, *Oszillations-matrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie Verlag, Berlin, 1960.
8 B. M. Green, On weakly sign-symmetric matrices, *Linear Algebra Appl.* 8:355–360 (1974).
9 D. Hershkowitz and A. Berman, Necessary conditions and a sufficient condition for the Fischer-Hadamard inequalities, *Linear and Multilinear Algebra*, to appear.
10 D. M. Kotelyanskiĭ, A property of sign symmetric matrices, *Uspekhi Mat. Nauk* 8:163–167 (1953); transl., *Amer. Math. Soc. Transl.* 27:19–24 (1963).
11 V. Mehrmann, On classes of matrices containing *M*-matrices, totally nonnegative and hermitian positive semidefinite matrices, Dissertation, Univ. Bielefeld, 1982.

Received 28 January 1983; revised 31 May 1983